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A THEOREM IN DETERMINANTS.

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1. The expression

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ a_4 & b_4 \end{vmatrix} \tag{1}$$

is obviously identically zero.

The identity corresponding to this in terms of determinants of the third order is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 & c_2 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix}$$

$$- \begin{vmatrix} a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \cdot \begin{vmatrix} a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} \equiv 0. \tag{2}$$

The following is a generalization of these identities for determinants of the nth order, viz., the sum of the expressions

$$\begin{array}{r} |1, 2, 3, 4, \ldots, n| \,.\, |n+1, n+2, n+3, \ldots, 2n| \\ (-1)^n\, |2, 3, 4, \ldots, n, n+1| \,.\, |1, n+2, n+3, \ldots, 2n| \\ |3, 4, \ldots, n, n+1, 1| \,.\, |2, n+2, n+3, \ldots, 2n| \\ (-1)^n\, |4, \ldots, n, n+1, 1, 2| \,.\, |3, n+2, n+3, \ldots, 2n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-1)^n\, |n+1, 1, 2, 3, \ldots, n-1| \,.\, |n, n+2, n+3, \ldots, 2n| \end{array} \tag{3}$$

is identically zero, where for convenience we put

$$|a_1^{(1)}, a_2^{(2)}, a_3^{(3)}, \ldots, a_n^{(n)}| = |1, 2, 3, \ldots, n|. \tag{4}$$

The identity of this sum with zero is made clear by expanding the determinants with regard to the elements of their left-hand columns; then on forming the products and summing, it appears that the coefficient of the general term is of the form

$$\begin{aligned} &|1, 2, 3, 4, \ldots, n-1| \,.\, |n, n+1, n+2, \ldots, 2(n-1)| \\ &+ (-1)^{n-1} |2, 3, 4, \ldots, n-1, 1| \,.\, |1, n+1, n+2, \ldots, 2(n-1)| \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ (-1)^{n-1} |n, 1, 2, \ldots, n-2| \,.\, |n-1, n+1, n+2, \ldots, 2(n-1)| ; \end{aligned} \tag{5}$$

treating this sum in the same manner the coefficient of its general term is found to be of the form (5) in which n is again decreased by unity; this process carried through $n-2$ steps leads to a coefficient of the form

$$|1, 2| \,.\, |3, 4| + |2, 3| \,.\, |1, 4| + |3, 1| \,.\, |2, 4| , \tag{6}$$

which is (1) in a different notation and identically zero; therefore, *a fortiori*, is the sum of the expressions (3) identically zero. Hence, we have

$$\Sigma\, |\alpha, \beta, \gamma, \delta, \ldots, \mu| \,.\, |\nu, n+2, n+3, \ldots, 2n| \equiv 0, \tag{7}$$

where

$$\alpha, \beta, \gamma, \delta, \ldots, \mu, \nu$$

takes successively the values of the sequence

$$1, 2, 3, 4, \ldots, n, n+1 ,$$

(or in fact, of any sequence of $n+1$ different indices) and of its cyclical permutations, and is always positive or alternately positive and negative, according as n is even or odd.

2. By noting that the volume of a solid having $n+1$ vertices in a space of n dimensions is equal to

$$\frac{1}{n!} \begin{vmatrix} x_1^{(1)} , & x_2^{(1)} , & \ldots, & x_n^{(1)} , & 1 \\ x_1^{(2)} , & x_2^{(2)} , & \ldots, & x_n^{(2)} , & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{(n+1)}, & x_2^{(n+1)}, & \ldots, & x_n^{(n+1)}, & 1 \end{vmatrix} , \tag{8}$$

$$x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}, \quad i = 1, 2, 3, \ldots, n+1 , \tag{9}$$

being the coordinates of the $n+1$ vertices, the above theorems may be interpreted geometrically.

Consider a system of $2(n+1)$ points in a space of n dimensions. These points form $\underline{|2(n+1)}/(\underline{|n+1})^2$ solids, each having $n+1$ vertices. Designating the points by the integers $1, 2, \ldots, n+1, n+2, \ldots, 2(n+1)$, and the volume of the solid whose $n+1$ vertices are $1, 2, 3, \ldots, n+1$ by $\{1, 2, 3, \ldots, n+1\}$, the theorem expressed by (7) becomes, in virtue of (8) and (5)

$$\begin{aligned}&\{1, 2, \ldots, n, n+1\} \,.\, \{n+2, n+3, \ldots, 2(n+1)\} \pm \{1, 2, \ldots, n, n+2\} \,.\\ &\qquad\qquad\qquad \{n+3, n+4, \ldots, 2(n+1), n+1\}\\ &+ \{1, 2, \ldots, n, n+3\} \,.\, \{n+4, \ldots, 2(n+1), n+1, n+2\} \pm \{1, 2, \ldots, n, n+4\} \,.\\ &\qquad\qquad\qquad \{n+5, \ldots, 2(n+1), n+1, n+2, n+3\}\\ &+ \,.\; .\\ &\pm \{1, 2, \ldots, n, 2(n+1)\} \,.\, \{n+1, n+2, n+3, \ldots, 2n+1\} \equiv 0, \qquad (10)\end{aligned}$$

the upper or lower signs being taken according as n is odd or even.

The volumes of these solids arrange themselves in pairs in $\underline{|2(n+1)}/2(\underline{|n+1})^2$ products to satisfy $[\underline{|2(n+1)}/2(\underline{|n+1})^2] - (n^2+1)$ independpendent identical relations of the form (10); the others are derived from (10) by permutation. The n^2 independent ratios of these binary products are the independent absolute invariants of the original system of $2(n+1)$ points by the general projective group of n dimensional space; the identities of the form (10) thus furnish a complete generalization of Clifford's statement of the general theory of anharmonics for the first three dimensions.

It may be further remarked by way of illustration that the identity (10) gives the well known relation connecting the distances of four points 1, 2, 3, 4 on a right line, viz.,

$$12 \,.\, 34 + 23 \,.\, 14 + 31 \,.\, 24 \equiv 0,$$

and the five independent identities* of the form

$$123 \,.\, 456 + 124 \,.\, 365 + 125 \,.\, 634 + 126 \,.\, 543 \equiv 0,$$

connecting the areas of triangles formed by six points 1, 2, 3, 4, 5, 6 in a plane.

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* Clifford is in error both as to the statement and the number of these identities on page 3 of volume 2 of the *Proceedings of the London Mathematical Society.*